(1) (20 points) Compute the general solution for each of the differential equations, (y is a function of x in each case).

(a) $5x^2y'' + 2xy' + 4y = 0, x > 0$

(b) $5y'' - y' - 4y = 2e^x.$

(a) Euler equation with indicial equation $5r(r-1)+2r+4 = 5r^2-3r+4 = 0$ which has roots

$$r = (3 \pm \sqrt{9-80})/10 = \frac{3}{10} \pm \frac{\sqrt{71}}{10}\mathbf{i}.$$

The general solution is:

$$y = C_1x^{3/10}\cos(\frac{\sqrt{71}}{10}\ln x) + C_2x^{3/10}\sin(\frac{\sqrt{71}}{10}\ln x).$$

(b) The characteristic of the homogeneous equation is $0 = 5r^2 - r - 4 = (5r+4)(r-1)$ with roots

$$r = -4/5, 1.$$

So $y_h = C_1e^{-4x/5} + C_2e^x$.

The associated root for $2e^x$ is 1 which is a root of the characteristic equation and so the test function is $Y_p = Axe^x$ with $Y_p' = Axe^x + Ae^x, Y_p'' = Axe^x + 2Ae^x$. So $2e^x = 5Y_p'' - Y_p' - 4Y_p = 9Ae^x$. Hence, $A = 2/9$. So the general solution is

$$y = C_1e^{-4x/5} + C_2e^x + 2xe^x/9.$$

(2)(10 points) A hanging spring is stretched 6 inches (= .5 feet) by a weight of 64 pounds.

(a) Set up the initial value problem (differential equation and initial conditions) which describes the motion, neglecting friction, when the weight is pulled down an additional foot and is then released and is subjected to an external force of $6\cos(\omega t)$. You need not solve the equation. (Recall that g, the acceleration due to gravity is 32 feet/second^2.)

(b) Write down the test function with the fewest terms which can be used to obtain a particular solution via the Method of Undetermined Coefficients when $\omega = 4$.

(c) Write down the test function with the fewest terms which can be used to obtain a particular solution via the Method of Undetermined Coefficients when ω is the *resonance frequency* that is, the natural frequency of the spring.

(a) $mg = w = 64, \quad m = 2\ \Delta L = 1/2$ and so $w = k\Delta L$ implies $k = 128$.
The equation is

$$2y'' + 0y' + 128y = 6\cos(\omega t) \quad \text{or} \quad y'' + 64y = 3\cos(\omega t)$$

with initial conditions $y(0) = -1, y'(0) = 0$.

The characteristic polynomial for the homogeneous equation is $r^2 + 64$ with roots $\pm 8\mathbf{i}$. Thus, the natural frequency of the spring is 8.

(b) $Y_p = A\cos(4t) + B\sin(4t)$.

(c) $Y_p = t[A\cos(8t) + B\sin(8t)]$.

(3) (18 points) For the differential equation:

$$(1 - 3x)y'' + (1 + 2x^3)y' + (4x^2 - 5)y \quad = \quad 0$$

Compute the recursion formula for the coefficients of the power series solution centered at $x_0 = 0$. Use it to compute the first four nonzero terms of the series for the solution with $y(0) = 4$ and $y'(0) = 0$ four terms including the first one].

$$
\begin{aligned}
1y'' &= \sum n(n-1)a_n x^{n-2} &&= \sum (k+2)(k+1)a_{k+2}x^k, \\
-3xy'' &= \sum -3n(n-1)a_n x^{n-1} &&= \sum -3(k+1)ka_{k+1}x^k, \\
1y' &= \sum na_n x^{n-1} &&= \sum (k+1)a_{k+1}x^k, \\
2x^3y' &= \sum n2a_n x^{n+2} &&= \sum 2(k-2)a_{k-2}x^k, \\
4x^2y &= \sum 4a_n x^{n+2} &&= \sum 4a_{k-2}x^k, \\
-5y &= \sum -5a_n x^n &&= \sum -5a_k x^k
\end{aligned}
$$

The recursion formula is

$$a_{k+2} = \frac{1}{(k+2)(k+1)}[3(k+1)ka_{k+1} - (k+1)a_{k+1} + 5a_k - 2(k-2)a_{k-2} - 4a_{k-2}].$$

or

$$a_{k+2} = \frac{1}{(k+2)(k+1)}[((3k-1)(k+1)a_{k+1} + 5a_k - 2ka_{k-2}].$$

$a_0 = 4.$
$a_1 = 0$
$k = 0 : a_2 = \frac{1}{2}[-a_1 + 5a_0 + 0] = \frac{1}{2}[20] = 10.$
$k = 1 : a_3 = \frac{1}{6}[4a_2 + 5a_1 - 2a_{-1}] = \frac{1}{6}[40 + 0 - 0] = \frac{20}{3}.$
$k = 2 : a_4 = \frac{1}{12}[15a_3 + 5a_2 - 4a_0] = \frac{1}{12}[100 + 50 - 16] = \frac{134}{12} = \frac{67}{6}.$

$$y = 4 + 10x^2 + \frac{20}{3}x^3 + \frac{67}{6}x^4 + \dots.$$

(4) (14 points) (a) State the definition of the Laplace transform and use it to compute to compute for a function $f(t) = e^{5t}$ the Laplace Transform $\mathcal{L}(f)(s)$.

(b) Compute the Laplace Transform $\mathcal{L}(y)$ for the solution of the initial value problem:

$$
\begin{aligned}
&2y'' - 5y' + 2y = 2e^{5t}, \\
&y(0) = 5 \quad \text{and} \quad y'(0) = -3.
\end{aligned}
$$

$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)\, dt$. With $f(t) = e^{5t}$,

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-(s-5)t}\, dt = -\frac{1}{s-5}e^{-st}|_0^\infty = \frac{1}{s-5}.$$

(b)$\mathcal{L}(y') = sL(y) - y(0), \mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0)$. and so

$$2\mathcal{L}(y'') - 5\mathcal{L}(y') + 2\mathcal{L}(y) = 2[s^2\mathcal{L}(y) - 5s + 3] - 5[s\mathcal{L}(y) - 5] + 2[\mathcal{L}(y)] = \frac{2}{s-5}.$$

$$(2s^2 - 5s + 2)\mathcal{L}(y) = \frac{2}{s-2} + 10s - 31.$$

$\mathcal{L}(y) = [\frac{2}{s-2} + 10s - 31] \div (2s^2 - 5s + 2)$.

(5) (18 points) For the differential equation:

$$(4x^2 - x^3)y'' - 2x^2y' + y \quad = \quad 0$$

The point $x = 0$ is a *regular singular point* . Compute the *associated Euler equation* and compute the recursion formula for the coefficients of the series solution centered at $x_0 = 0$ which is associated with the larger root. Compute the first three non-zero terms of the series (not just the coefficients) with $a_0 = 16$. (22 points)

The associated Euler equation is $4x^2y'' + 0xy' + y = 0$ with indicial equation $4r(r-1) + 1 = (2r-1)^2$. The larger (only) root is $r = \frac{1}{2}$.

$$4x^2y'' = \sum 4(n+r)(n+r-1)a_nx^{n+r} = \sum 4(k+r)(k+r-1)a_kx^{k+r}.$$
$$-x^3y'' = \sum -(n+r)(n+r-1)a_nx^{n+r+1} = \sum -(k+r-1)(k+r-2)a_{k-1}x^{k+r}.$$
$$-2x^2y' = \sum -2(n+r)a_nx^{n+r+1} = \sum -2(k+r-1)a_{k-1}x^{k+r}.$$
$$y = \sum a_nx^{n+r} = \sum a_kx^{k+r}.$$

$$(4(k+r)(k+r-1) + 1)a_k = (2(k+r) - 1)^2a_k =$$
$$[(k+r-1)(k+r-2) + 2(k+r-1)]a_{k-1} = [(k+r-1)(k+r)]a_{k-1}.$$

With $r = \frac{1}{2} : 4k^2a_k = \frac{(2k-1)(2k+1)}{4})a_{k-1}$.
and the recursion formula is:

$$a_k = \frac{1}{16k^2}[4k^2 - 1]a_{k-1}.$$

With $a_0 = 16 :$
$k = 1 : a_1 = 3, \quad k = 2 : a_2 = \frac{45}{64}$.

$$y = x^{1/2}[16 + 3x + \frac{45}{64}x^2 + ...$$

(6) (20 points) (a) Compute the Fourier series for the function $f(x) = x^2$ on the interval $[-4, 4]$.

(b) Compute the solution $u(t, x)$ for the partial differential equation on the interval $[0, 4]$:

$$u_t = 9u_{xx} \quad \text{with}$$
$$u(t,0) = u(t,4) = 0 \quad \text{for } t > 0 \qquad \text{(boundary conditions)}$$
$$u(0,x) = 7\sin(\pi x) - \sin(4\pi x) \quad \text{for } 0 < x < 4 \qquad \text{(initial conditions)}$$

(a) This is an even function and so $b_n = 0$ for all n.

$$a_0 = \frac{2}{4}\int_0^4 x^2 dx = 32/3.$$

For $n > 0$,

$$a_n = \frac{2}{4}\int_0^4 x^2 \cos(\frac{n\pi x}{4})\ dx = \frac{1}{2}\times$$
$$[(x^2)(\frac{4}{n\pi}\sin(\frac{n\pi x}{4})) - (2x)(-(\frac{4}{n\pi})^2\cos(\frac{n\pi x}{4}))$$
$$+(2)((-\frac{4}{n\pi})^3\sin(\frac{n\pi x}{4}))]_0^4 =$$
$$\frac{1}{2}\times[0 + (8)(\frac{4}{n\pi})^2(-1)^n - 0].$$

$$a_n = (-1)^n)(\frac{64}{(n\pi)^2}).$$

$$x^2 = \frac{16}{3} + \sum_{n=1}^{\infty}(\frac{8}{n\pi})^2(-1)^n\cos(\frac{n\pi x}{4}).$$

(b) $\alpha = 3$ and $L = 4$.

$$u(t,x) = 7exp(-(3\pi)^2 t)\sin(\pi x) - exp(-(12\pi)^2 t)\sin(4\pi x)$$

Good Luck. Remember to show your work.