Math 39100 K (19535) - Lectures 02

Ethan Akin
Email: eakin@ccny.cuny.edu

Fall, 2026

Contents

Second Order Linear Equations, B & D Chapter 4

- Second Order Linear Homogeneous Equations with Constant Coefficients
- Polar Form of Complex Numbers; The Euler Identity
- Reduction of Order
- Nonhomogeneous Equations
- Undetermined Coefficients
- Question

Illustration

Higher Order Equations with Constant Coefficients, Chapter 4

- Undetermined Coefficients for Higher Order Equations

Second Order Linear Equations, Again

- Variation of Parameters
- Spring Problems
- Ordinary Point Problems, B & D Sections 5.2, 5.3

Second Order Differential Equations

A general second order ode (= ordinary differential equation) is of the form $y'' = F(x, y, y')$ or $\frac{d^2y}{dt^2} = F(t, y, \frac{dy}{dt})$.

For second order equations, we require two integrations to get back to the original function. So the *general solution* has two separate, arbitrary constants. An IVP in this case is of the form:

$$y'' = F(t, y, y'), \quad \text{with} \quad y(t_0) = y_0, \ y'(t_0) = y_0'.$$

Interpreting y' as the velocity and y'' as the acceleration, we thus determine the constants by specifying the initial position and the initial velocity with t_0 regarded as the initial time (usually $= 0$).

The Fundamental Existence and Uniqueness Theorem says that an IVP has a unique solution.

The General Solution

The *general solution* of the equation $y'' = F(t, y, y')$ is of the form $y(t, C_1, C_2)$ with two arbitrary constants. Supposing that these are solutions for every choice of C_1, C_2, how do we now that we have all the solutions? The answer is that we have all solutions if we can solve every IVP. To be precise:

THEOREM: If $y'' = F(t, y, y')$ is an equation to which the Fundamental Theorem applies and $y(t, C_1, C_2)$ is a solution for every C_1, C_2, then every solution is one of these provided that for every pair of real numbers A, B, we can find C_1, C_2 so that $y(t_0, C_1, C_2) = A$ and $y'(t_0, C_1, C_2) = B$.

PROOF: If $z(t)$ is any solution then let $A = z(t_0)$ and $B = z'(t_0)$, and choose C_1, C_2 so that $y(t, C_1, C_2)$ solves this IVP. Then $y(t, C_1, C_2)$ and $z(t)$ solve the same IVP. This means that $z(t) = y(t, C_1, C_2)$.

Linear Equations and Linear Operators

We will be studying exclusively second order, *linear* equations. These are of the form

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = f(t), \quad \text{or} \quad y'' + py' + qy = f.$$

Just as in the first order linear case, if we see $Ay'' + By' + Cy = D$ with A, B, C, D functions of t we can obtain the above form by dividing by A. When A, B and C are constants we often won't bother.

The equation is called *homogeneous* when $f = 0$. It has *constant coefficients* when p and q are constant functions of t.

These are studied by using *linear operators*. An *operator* is a function $\mathcal{L}$ whose input and output are functions. For example, $\mathcal{L}(y) = yy'$. If $y = e^x$ then $\mathcal{L}(y) = e^{2x}$ and $\mathcal{L}(\sin)(x) = \sin(x)\cos(x)$.

An operator $\mathcal{L}$ is a linear operator when it satisfies *linearity* or *The Principle of Superposition*.

$$\mathcal{L}(Cy_1 + y_2) = C\mathcal{L}(y_1) + \mathcal{L}(y_2).$$

Given functions p, q we define $\mathcal{L}(y) = y'' + py' + qy$. Observe that:

$$\begin{array}{rcccc} C \times (y_1'' & + & py_1' & + & qy_1) \\ + \quad y_2'' & + & py_2' & + & qy_2 \\ \hline (Cy_1 + y_2)'' + p(Cy_1 + y_2)' + q(Cy_1 + y_2) \end{array}$$

So the homogeneous equation $y'' + py' + qy = 0$ can be written as $\mathcal{L}(y) = 0$.

General Solution of the Homogeneous Equation, Section 3.2

If y_1 and y_2 are solutions of the homogeneous equation $\mathcal{L}(y) = 0$ then by linearity $C_1y_1 + C_2y_2$ are solutions for every choice of constants C_1 and C_2, because $\mathcal{L}(C_1y_1 + C_2y_2) = C_1\mathcal{L}(y_1) + C_2\mathcal{L}(y_2) = C_10 + C_20 = 0$.

Notice there are two arbitrary constants. This is the general solution provided we can solve every IVP. So given numbers A, B we want to find C_1, C_2 so that

$$C_1y_1(t_0) + C_2y_2(t_0) = A,$$
$$C_1y_1'(t_0) + C_2y_2'(t_0) = B.$$

Cramer's Rule says that this has a solution provided the coefficient determinant

$$\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}$$ is nonzero.

The Wronskian

Given two functions y_1, y_2 the *Wronskian* is defined to be the function:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'.$$

Two functions are called *linearly dependent* when there are constants C_1, C_2 not both zero such that $C_1 y_1 + C_2 y_2 \equiv 0$ and so when one of the two functions is a constant multiple of the other. If $y_2 = C y_1$ then $W(y_1, y_2) \equiv 0$. The converse is almost true:

$W/y_1^2 = \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = (\frac{y_2}{y_1})'$ and so if W is identically zero the ratio $\frac{y_2}{y_1}$ is some constant C and so $y_2 = C y_1$.

On the other hand, $y_1(t) = t^3$ and $y_2(t) = |t^3|$ have $W \equiv 0$ but are not linearly dependent. What is the problem? Hint: look for a division by zero.

Now suppose the Wronskian vanishes at a single point t_0. If $y_1'(t_0) = y_2'(t_0) = 0$ then we can pick C_1, C_2, not both zero, such that with $z = C_1y_1 + C_2y_2$ is zero at t_0 and of course $z'(t_0) = C_1y_1'(t_0) + C_2y_2'(t_0) = 0$. Otherwise, choose $C_1 = y_2'(t_0)$ and $C_1 = -y_1'(t_0)$. Check that with $z = C_1y_1 + C_2y_2$ we have

$$z(t_0) = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0) = W(y_1, y_2)(t_0) = 0,$$
$$z'(t_0) = y_1'(t_0)y_2'(t_0) - y_2'(t_0)y_1'(t_0) = 0.$$

If y_1, y_2 are solutions of the homogeneous equation $y'' + py' + qy = 0$ then $z = C_1y_1 + C_2y_2$ is a solution with zero initial conditions. This means that $z \equiv 0$ and so y_1 and y_2 are linearly dependent.

A pair y_1, y_2 of solutions of the homogeneous equation $y'' + py' + qy = 0$ is called a *fundamental pair* if the Wronskian does not vanish. The general solution is then $C_1y_1 + C_2y_2$ because we can solve every IVP.

Abel's Theorem

Abel's Theorem shows that the Wronskian of two solutions y_1, y_2 of $y'' + py' + qy = 0$ satisfies a first order linear ode:

$$W' = y_1y_2'' - y_2y_1'' + y_1'y_2' - y_2'y_1' = y_1y_2'' - y_2y_1''$$

$$\begin{array}{rccccc} q \times & y_1y_2 & - & y_2y_1 & = & 0 \\ p \times & y_1y_2' & - & y_2y_1' & = & W \\ 1 \times & y_1y_2'' & - & y_2y_1'' & = & W' \end{array}$$

The left side adds up to $y_1 0 - y_2 0 = 0$ and so $0 = W' + pW$ or $\frac{dW}{dt} + pW = 0$.

This is variables separable with solution $W = C \times e^{-\int p(t)dt}$.

Since the exponential is always positive, $W \equiv 0$ if $C = 0$ and otherwise W is never zero.

Homogeneous, Constant Coefficients, Sections 3.1, 3.3

The derivative itself is an example of a linear operator: $D(y) = y'$. For a linear operator $\mathcal{L}$ a function y is an *eigenvector* with *eigenvalue* r when $\mathcal{L}(y) = ry$. For the operator D, an eigenvector y satisfies $y' = D(y) = ry$. This is exponential growth with growth rate r and so the solutions are constant multiples of $y = e^{rt}$. Thus, such exponential functions have a special role to play.

Example: $y'' - y' - 6y = 0$. So that $\mathcal{L}(y) = y'' - y' - 6y$.

We can write
$\mathcal{L} = (D^2 - D - 6)(y) = (D+2)(D-3)(y) = (D-3)(D+2)(y)$.

So if $(D-3)(y) = 0$ or $(D+2)(y) = 0$, then $\mathcal{L}(y) = 0$.
$(D-3)(y) = 0$ when $y' = 3y$ and so when $y = e^{3t}$.
$(D+2)(y) = 0$ when $y = e^{-2t}$.
So $y_1 = e^{3t}$ and $y_2 = e^{-2t}$ are solutions of $y'' - y' - 6y = 0$.

A second order, linear, homogeneous equation with constant coefficients is of the form $Ay'' + By' + Cy = 0$ with A, B, C constants and $A \neq 0$.

We look for a solution of the form $y = e^{rt}$. Substituting and factoring we get

$$(Ar^2 + Br + C)e^{rt} = 0.$$

Since the exponential is always positive, e^{rt} is a solution precisely when r satisfies the characteristic equation $Ar^2 + Br + C = 0$.

The quadratic equation $Ar^2 + Br + C = 0$ has roots

$$r = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

For the equation $Ar^2 + Br + C = 0$ the nature of the two roots

$$r = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

depends on the discriminant $B^2 - 4AC$.

CASE 1: The simplest case is when $B^2 - 4AC > 0$. There are then two distinct real roots r_1 and r_2. This gives us two special solutions $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ with the Wronskian.

$$W = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1) e^{(r_1 + r_2)t} \neq 0.$$

The general solution is then $C_1 e^{r_1 t} + C_2 e^{r_2 t}$.

CASE 2: If $B^2 - 4AC < 0$ then the roots are a complex conjugate pair $a \pm \mathbf{i}b = \frac{-B}{2A} \pm \mathbf{i}\frac{\sqrt{|B^2 - 4AC|}}{2A}$. This requires a consideration of complex numbers.

Polar Form of Complex Numbers; The Euler Identity

We represent the complex number $z = a + \mathbf{i}b$ as a vector in $\mathbb{R}^2$ with x-coordinate a and y-coordinate b. Addition and multiplication by real scalars is done just as with vectors. For multiplication $(a + \mathbf{i}b)(c + \mathbf{i}d) = (ac - bd) + \mathbf{i}(ad + bc)$. The *conjugate* is $\bar{z} = a - b\mathbf{i}$ and $z\bar{z} = a^2 + b^2 = |z|^2$ which is positive unless $z = 0$.

Multiplication is also dealt with by using the *polar* form of the complex number. This requires a bit of review.

Recall from Math 203 the three important Maclaurin series:

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + \frac{t^7}{7!} + \ldots$$

$$\cos(t) = 1 - \frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots$$

$$\sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots$$

The exponential function is the first important example of a function f on $\mathbb{R}$ which is neither *even* with $f(-t) = f(t)$ nor *odd* with $f(-t) = -f(t)$ nor a mixture of even and odd functions in an obvious way like a polynomial. It turns out that any function f on $\mathbb{R}$ can be written as the sum of an even and an odd function by writing

$$f(t) \quad = \quad \frac{f(t) + f(-t)}{2} \; + \; \frac{f(t) - f(-t)}{2}.$$

You have already seen this for $f(t) = e^t$

$$\begin{aligned}
e^{-t} &= 1 - t + \frac{t^2}{2} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \frac{t^6}{6!} - \frac{t^7}{7!} + \ldots \\
\cosh(t) = \frac{e^t + e^{-t}}{2} &= 1 + \frac{t^2}{2} + \frac{t^4}{4!} + \frac{t^6}{6!} + \ldots \\
\sinh(t) = \frac{e^t - e^{-t}}{2} &= t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \ldots
\end{aligned}$$

Now we use the same trick but with $e^{\mathbf{i}t}$

$$
\begin{aligned}
e^{\mathbf{i}t} &= 1 + \mathbf{i}t - \frac{t^2}{2} - \mathbf{i}\frac{t^3}{3!} + \frac{t^4}{4!} + \mathbf{i}\frac{t^5}{5!} - \frac{t^6}{6!} - \mathbf{i}\frac{t^7}{7!} + \ldots \\
e^{-\mathbf{i}t} &= 1 - \mathbf{i}t - \frac{t^2}{2} + \mathbf{i}\frac{t^3}{3!} + \frac{t^4}{4!} - \mathbf{i}\frac{t^5}{5!} - \frac{t^6}{6!} + \mathbf{i}\frac{t^7}{7!} + \ldots \\
\frac{e^{\mathbf{i}t} + e^{-\mathbf{i}t}}{2} &= 1 - \frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots \\
\frac{e^{\mathbf{i}t} - e^{-\mathbf{i}t}}{2} &= \mathbf{i}[t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots]
\end{aligned}
$$

So the even part of $e^{\mathbf{i}t}$ is $\cos(t)$ and the odd part is $\mathbf{i}\sin(t)$. Adding them we obtain *Euler'sIdentity* and its conjugate version.

$$
\begin{aligned}
e^{\mathbf{i}t} &= \cos(t) + \mathbf{i}\sin(t) \\
e^{-\mathbf{i}t} &= \cos(t) - \mathbf{i}\sin(t)
\end{aligned}
$$

Substituting $t = \pi$ we obtain $e^{\mathbf{i}\pi} = -1$ and so $e^{\mathbf{i}\pi} + 1 = 0$.

Now we start with a complex number z in rectangular form $z = x + \mathbf{i}y$ and convert to polar coordinates with $x = r\cos(\theta), y = r\sin(\theta)$ so that $r^2 = x^2 + y^2 = z\bar{z}$. The length r is called the *magnitude* of z. The angle θ is called the *argument* of z. We obtain

$$z \quad = \quad x + \mathbf{i}y \quad = \quad r(\cos(\theta) + \mathbf{i}\sin(\theta)) \quad = \quad re^{\mathbf{i}\theta}.$$

Now we return to CASE 2 with the pair of roots $r = a \pm \mathbf{i}b$. We obtain two complex solutions

$$e^{(a+\mathbf{i}b)t} = e^{at}(\cos(bt) + \mathbf{i}\sin(bt)),$$
$$e^{(a-\mathbf{i}b)t} = e^{at}(\cos(bt) - \mathbf{i}\sin(bt))$$

When we add these two solutions and divide by 2 we obtain the real solution $e^{at}\cos(bt)$ When we subtract and divide by $2\mathbf{i}$ we obtain the real solution $e^{at}\sin(bt)$.

Thus, when the roots are $r = a \pm \mathbf{i}b$, we have the two solutions $y_1 = e^{at}\cos(bt)$ and $y_2 = e^{at}\sin(bt)$ with the Wronskian

$$W = \begin{vmatrix} e^{at}\cos(bt) & e^{at}\sin(bt) \\ ae^{at}\cos(bt) - be^{at}\sin(bt) & ae^{at}\sin(bt) + be^{at}\cos(bt) \end{vmatrix}$$

This is be^{2at} which is not zero because the imaginary part b is not zero. Thus we have so far:

CASE 1 ($B^2 - 4AC > 0$, Two distinct real roots r_1, r_2) The general solution is $C_1 e^{r_1 t} + C_2 e^{r_2 t}$

CASE 2 ($B^2 - 4AC < 0$, Complex conjugate pair of roots $a \pm \mathbf{i}b$) The general solution is $C_1 e^{at}\cos(bt) + C_2 e^{at}\sin(bt)$.

Example 3.3/ BD 19, BDM 13: $y'' - 2y' + 5y = 0$, $y(\pi/2) = 0,\ y'(\pi/2) = 3.$

The characteristic equation is $r^2 - 2r + 5 = 0$ with roots $r = \frac{+2 \pm \sqrt{4-20}}{2} = 1 \pm 2\mathbf{i}$.

So the general solution is:

$$y = C_1 e^t \cos(2t) + C_2 e^t \sin(2t).$$

When $t = \pi/2, \sin(2t) = 0, \cos(2t) = -1.$

$$
\begin{aligned}
y &= C_1 e^t \cos(2t) + C_2 e^t \sin(2t), \\
y' &= C_1[e^t \cos(2t) - 2e^t \sin(2t)] + \\
&\qquad C_2[e^t \sin(2t) + 2e^t \cos(2t)].
\end{aligned}
$$

At $t = \pi/2, \quad 0 = C_1[-e^{\pi/2}] + C_2[0]$,
$3 = C_1[-e^{\pi/2}] + C_2[-2e^{\pi/2}]$.
So $C_1 = 0, C_2 = -(3/2)e^{-\pi/2}$ and so
$y = -(3/2)e^{-\pi/2} e^t \sin(2t)$.

We looked at $y'' - 3y = 0$ which had, we saw, general solution

$$y = C_1 e^{\sqrt{3}t} + C_2 e^{-\sqrt{3}t}.$$

Now look at $y'' + 3y = 0$ with characteristic equation
$r^2 + 3 = 0$.
It has conjugate roots $\pm\sqrt{3}\mathbf{i}$ and so the general solution is:

$$y = C_1 \cos(\sqrt{3}t) + C_2 \sin(\sqrt{3}t).$$

Reduction of Order, Section 3.4

There remains CASE 3: If $B^2 - 4AC = 0$ there is a single (real) root $r = \frac{-B}{2A}$, "repeated twice". So we have so far in that case only one solution e^{rt} and we need a second independent solution.

For a general second order, linear homogeneous equation $y'' + py' + qy = 0$ with constant coefficients or not, we need two independent solutions y_1, y_2 to get the general solution $C_1 y_1 + C_2 y_2$.

Suppose we have one solution y_1. We look for another solution of the form $y_2 = uy_1$, with u not a constant. Substitute:

$$\begin{array}{rrcl} q \times & y_2 & = & uy_1 \\ p \times & y_2' & = & uy_1' + u'y_1 \\ 1 \times & y_2'' & = & uy_1'' + 2u'y_1' + u''y_1 \\ \\ & 0 & = & 0 + u'(py_1 + 2y_1') + u''y_1 \end{array}$$

The equation $(y_1)u'' + (py_1 + 2y_1')u' = 0$ has no variable u and so by letting $v = u', v' = u''$ we obtain the first order equation $(y_1)v' + (py_1 + 2y_1')v = 0$ which variables separable with solution $v = exp(-\int\ p + 2(y_1'/y_1))$. Notice that $u' = v$ is not zero and so when we integrate to get u it is not a constant and so $y_2 = uy_1$ together with y_1 forms a fundamental pair of solutions.

Now we return to CASE 3 where the quadratic equation $Ar^2 + Br + C = 0$ has a repeated root $r = r^*$. This means that $Ar^2 + Br + C = A(r - r^*)^2$ and so $B = -2Ar^*$ and $C = A(r^*)^2$.

The ode is $y'' - 2r^*y' + (r^*)^2 y = 0$ with a solution $y_1 = e^{r^*t}$. We look for $y_2 = ue^{r^*t}$.

$$\begin{array}{rrcl} (r^*)^2 \times & y_2 & = & ue^{r^*t} \\ -2r^* \times & y_2' & = & ur^*e^{r^*t} \quad + u'e^{r^*t} \\ 1 \times & y_2'' & = & u(r^*)^2 e^{r^*t} + 2u'r^*e^{r^*t} + u''e^{r^*t} \end{array}$$

$$0 = 0 + 0 + u''e^{r^*t}$$

So $v' = u'' = 0, v = 1$ and $u = t$. So the general solution for CASE 3 is $C_1 e^{r^*t} + C_2 te^{r^*t}$.

This completes the story for second order, linear homogeneous equations with constant coeeficients.

CASE 1 ($B^2 - 4AC > 0$, Two distinct real roots r_1, r_2) The general solution is $C_1 e^{r_1 t} + C_2 e^{r_2 t}$

CASE 2 ($B^2 - 4AC < 0$, Complex conjugate pair of roots $a \pm \mathbf{i}b$) The general solution is $C_1 e^{at} \cos(bt) + C_2 e^{at} \sin(bt)$.

CASE 3 ($B^2 - 4AC = 0$, single repeated, real root r^*) The general solution is $C_1 e^{r^* t} + C_2 t e^{r^* t}$.

In CASE 3, we used the method of Reduction of Order, but we simply write down the solution. However, for nonconstant coefficients we will require the method itself.

Example 3.4/ BD11, BDM9 :
$9y'' - 12y' + 4y = 0, \quad y(0) = 2, y'(0) = -1$

Characteristic Equation $0 = 9r^2 - 12r + 4 = (3r - 2)(3r - 2)$
$r = 2/3$ repeated twice.

The general solution is

$$y = C_1 e^{(2/3)t} + C_2 t e^{(2/3)t}.$$

$$\begin{aligned} y &= C_1 e^{(2/3)t} + C_2 t e^{(2/3)t}, \\ y' &= [(2/3)C_1 + C_2]e^{(2/3)t} + (2/3)C_2 t e^{(2/3)t}, \end{aligned}$$

$y(0) = 2, y'(0) = -1$ and so

$$2 = C_1, \quad -1 = (2/3)C_1 + C_2,$$

And so $-(7/3) = C_2$.

The solution is $y = [2 - (7/3)t]e^{(2/3)t}$.

Example 3.4/BD28 : $(x-1)y'' - xy' + y = 0, x > 1$ with $y_1(x) = e^x$.

$$\begin{array}{rrcl} 1 \times & y_2 & = & ue^x \\ -x \times & y_2' & = & ue^x + u'e^x \\ (x-1) \times & y_2'' & = & ue^x + 2u'e^x + u''e^x \end{array}$$

$$0 = 0 + u'(x-2)e^x + u''(x-1)e^x$$

$(x-1)v' = -(x-2)v$, and so $\int \frac{dv}{v} = \int -1 + \frac{1}{x-1} dx$.

$u' = v = (x-1)e^{-x}$, and so $u = -xe^{-x}$ and $y_2 = uy_1 = -x$.

Using the Wronskian

As one student pointed out, there is an alternative way of obtaining a second solution by using Abel's Theorem. Recall that it says that if W is the Wronskian for a pair of solutions y_1, y_2 of the second order, homogeneous, linear equation $y'' + py' + qy = 0$ then

$$W = C \cdot exp[-\int p(t)dt].$$

Suppose we have a solution y_1. We look for a second solution y_2 such that $W(y_1, y_2) = exp[-\int p(t)dt]$. We choose $C = 1$ because we are looking for a single solution y_2. So y_2 satisfies the first order linear equation

$$y_1 y_2' - y_1' y_2 = exp[-\int p(t)dt].$$

As an example the student chose: $x^2 y'' - 7xy' + 16y = 0$ with $y_1 = x^4$.

$$\begin{aligned} 16 \times \ y_2 &= ux^4 \\ -7x \times \ y_2' &= u4x^3 + u'x^4 \\ x^2 \times \ y_2'' &= u12x^2 + u'8x^3 + u''x^4 \end{aligned}$$

$$0 = 0 + u'x^5 + u''x^6$$

$x^6 v' = -x^5 v$, and so $\int \frac{dv}{v} = \int -\frac{dx}{x}$.

$\ln v = -\ln x$ and so $u' = v = x^{-1}$,

$$u = \ln x, \qquad y_2 = uy_1 = x^4 \ln x$$

.

Instead we use Abel's Theorem applied to: $y'' - \frac{7}{x}y' + \frac{16}{x^2}y = 0$.

$$x^4 y_2' - 4x^3 y_2 = exp[7 \ln x] = x^7,$$

$$y_2' - \frac{4}{x} y_2 = x^3.$$

$$\mu = x^{-4}, [x^{-4} y_2]' = x^{-1}.$$

$$x^{-4} y_2 = \ln x, \qquad y_2 = x^4 \ln x.$$

Second Order Linear Nonhomogeneous Equations

Consider now a second order linear equation which is not homogeneous $y'' + py' + qy = f$ where p, q and f are functions of t.

Recall the linear operator $\mathcal{L}(y) = y'' + py' + qy$. To solve the homogeneous equation $\mathcal{L}(y) = 0$ we saw that we need a pair of independent solutions y_1, y_2. Then the general solution of the homogeneous equation is $y_h = C_1 y_1 + C_2 y_2$. To solve the equation $\mathcal{L}(y) = f$ we need just one solution, *particular solution* y_p so that $\mathcal{L}(y_p) = f$. Then the general solution is $y_g = y_h + y_p = C_1 y_1 + C_2 y_2 + y_p$. Notice first

$$\mathcal{L}(C_1 y_1 + C_2 y_2 + y_p) = C_1 \mathcal{L}(y_1) + C_2 \mathcal{L}(y_2) + \mathcal{L}(y_p) = 0 + 0 + f = f.$$

So for every choice of C_1 and C_2, y_g is a solution.

To show that by varying the two arbitrary constants y_g gives us all solutions, it is enough to show that we can solve every initial value problem. That is, given A, B we have to choose C_1, C_2 to satisfy the initial conditions $y_g(t_0) = A, y_g'(t_0) = B$. This means that we want to solve:

$$\begin{aligned} C_1 y_1(t_0) + C_2 y_2(t_0) + y_p(t_0) &= A, \\ C_1 y_1'(t_0) + C_2 y_2'(t_0) + y_p'(t_0) &= B. \end{aligned}$$

But this is the same as:

$$\begin{aligned} C_1 y_1(t_0) + C_2 y_2(t_0) &= A - y_p(t_0), \\ C_1 y_1'(t_0) + C_2 y_2'(t_0) &= B - y_p'(t_0). \end{aligned}$$

This has a solution because the Wronskian $W(y_1, y_2)(t_0) \neq 0$. Since every IVP has a solution of the form y_g, we have all solutions.

Nonhomogeneous Equations: Undetermined Coefficients, Section 3.5

Our first method for finding the particular solution applies when the homogeneous equation $Ay'' + By' + Cy = 0$ has constant coefficients the *forcing function* is a sum so that each term has an associated root according to the following table.

$$\begin{array}{rcl} poly(t) & \Rightarrow & \text{root equals } 0, \\ poly(t)e^{rt} & \Rightarrow & \text{root equals } r, \\ \begin{array}{r} poly(t)e^{at}\cos(bt) \\ poly(t)e^{at}\sin(bt) \end{array} & \Rightarrow & \text{root equals } a \pm \mathbf{i}b. \end{array}$$

The key is that when you take the derivative of expressions associated with a particular root (or conjugate root pair) you get expressions associated with the same root and with the polynomial in front of no higher degree.

To solve $Ay'' + By' + Cy = f$ with forcing function $f(t)$ we find the *Test Function (First Version)* $Y_p^1(t)$ in two steps:

STEP 1: Find the associated root for each term and group according to the associated roots.

STEP 2a: For the real associated root r (including $r = 0$), you use

$$(A_n t^n + A_{n-1} t^{n-1} + \ldots A_1 t + A_0) e^{rt}$$

where n is the highest power of t attached to any e^{rt} in $r(t)$.

STEP 2b: For the complex pair $a \pm \mathbf{i}b$, you use

$$(A_n t^n + A_{n-1} t^{n-1} + \ldots A_1 t + A_0) e^{at} \cos(bt) +$$
$$(B_n t^n + B_{n-1} t^{n-1} + \ldots B_1 t + B_0) e^{at} \sin(bt)$$

where n is the highest power of t attached to any $e^{at}\cos(bt)$ or $e^{at}\sin(bt)$ in $r(t)$.
IMPORTANT: You have to include both the sines and the cosines with that highest power applied to both.

To get the final version of the test function we need a preliminary step.

STEP 0: Solve the homogeneous equation $Ay'' + By' + Cy = 0$. In the process, you get the roots of the characteristic equation $Ar^2 + Br + C = 0$.

STEP 3: If a root from the homogeneous equation occurs among the associated roots then the corresponding expression in Y_p^1 is multiplied by the just high enough power of t so that the sum includes no solution of the homogeneous equation. After this adjustment we have the final Test Function Y_p.

STEP 4: Substitute Y_p into the equation and equate coefficients to determine the unknown coefficients to obtain the particular solution y_p.

The general solution is then $y_g = y_h + y_p$. If there are initial conditions, then substitute to determine C_1 and C_2 after y_p has been obtained.

Example: $f(t) =$
$t^3 + 2t - 5 + 7\cos(3t) - (t^2+1)e^{2t} - e^{2t}\sin(3t) - t^2\sin(3t)$.

STEP 1: The associated roots are as follows:

$$\begin{aligned} t^3 + 2t - 5 &\Rightarrow \text{root equals } 0, \\ 7\cos(3t) - t^2\sin(3t) &\Rightarrow \text{root equals } 0 \pm 3\mathbf{i}, \\ -(t^2+1)e^{2t} &\Rightarrow \text{root equals } 2, \\ -e^{2t}\sin(3t) &\Rightarrow \text{root equals } 2 \pm 3\mathbf{i}. \end{aligned}$$

STEP 2: The first version of the test function, $Y_p^1(t)$ is

$$\begin{gathered} [At^3 + Bt^2 + Ct + D] + \\ [(Et^2 + Ft + G)\cos(3t) + (Ht^2 + It + J)\sin(3t)] + \\ [(Kt^2 + Lt + M)e^{2t}] + \\ [Ne^{2t}\cos(3t) + Pe^{2t}\sin(3t)]. \end{gathered}$$

The final form of Y_p depends on the left side of the equation.
Example A: $y'' - 2y' = f(t)$, with characteristic equation $r^2 - 2r = 0$ and so with roots $0, 2$.
STEP 0: $y_h = C_1 + C_2 e^{2t}$.
STEP 3: In Y_p^1 the terms D and Me^{2t} are solutions of the homogeneous equation. When we apply $\mathcal{L}(y) = y'' - 3y'$, these terms drop out, with D and M gone. For the remaining 13 unknowns we get 15 equations and usually no solution. We multiply each of the corresponding blocks by t so that Y_p is

$$[t(At^3 + Bt^2 + Ct + D)]+$$
$$[(Et^2 + Ft + G)\cos(3t) + (Ht^2 + It + J)\sin(3t)]+$$
$$[t(Kt^2 + Lt + M)e^{2t}]+$$
$$[Ne^{2t}\cos(3t) + Pe^{2t}\sin(3t)].$$

STEP 4: Substitute Y_p into the equation $\mathcal{L}(y) = f$ and solve for the 15 unknowns to obtain the particular solution y_p. The general solution is then $y_g = y_h + y_p$.

Example B: $y'' - 4y' + 4y = f(t)$, with characteristic equation $r^2 - 4r + 4 = 0$ and so with roots $2, 2$.

STEP 0: $y_h = C_1 e^{2t} + C_2 t e^{2t}$.

STEP 3: In Y_p^1 the terms $Lte^{2t} + Me^{2t}$ are solutions of the homogeneous equation. Multiplying by t will not suffice as we then obtain Mte^{2t} which is still a solution of the homogeneous equation. Multiply by t again to get for Y_p:

$$[At^3 + Bt^2 + Ct + D] +$$
$$[(Et^2 + Ft + G)\cos(3t) + (Ht^2 + It + J)\sin(3t)] +$$
$$[t^2(Kt^2 + Lt + M)e^{2t}] +$$
$$[Ne^{2t}\cos(3t) + Pe^{2t}\sin(3t)].$$

Example C: $y'' + 9y = f(t)$ with characteristic equation $r^2 + 9 = 0$ and so with roots $\pm 3\mathbf{i}$.

STEP 0: $y_h = C_1 \cos(3t) + C_2 \sin(3t)$.

STEP 3: Now it is $G \cos(3t) + J \sin(3t)$ which case the problem. Multiplying by t we get for Y_p

$$[At^3 + Bt^2 + Ct + D] +$$
$$[t(Et^2 + Ft + G)\cos(3t) + t(Ht^2 + It + J)\sin(3t)] +$$
$$[(Kt^2 + Lt + M)e^{2t}] +$$
$$[Ne^{2t}\cos(3t) + Pe^{2t}\sin(3t)].$$

Question:

$$\begin{cases} (a) & y'' - y' - 2y \\ (b) & y'' - y' + 2y \\ (c) & 4y'' - 4y' + y \end{cases} =$$

$$t^2 + 7te^{2t} - e^{t/2}\sin((\sqrt{7}/2)t) - t^2e^{t/2} + 1.$$

(0) Solve each of the three homogeneous equations.
(1) Block together the terms with the same associated root.
(2) Obtain the first version of the test function $Y^1(t)$ by using the highest power ot t for each block. Remember to include both sines and cosines.
(3) Adjust any block whose associated root is a root of the homogeneous equation to get the test function $Y(t)$.

(0 - a) Characteristic equation $r^2 - r - 2 = (r-2)(r+1) = 0$ with roots $2, -1$. So $y_h = C_1 e^{2t} + C_2 e^{-t}$.
(0 - b) Characteristic equation $r^2 - r + 2 = 0$ with roots $\frac{1}{2} \pm \frac{\sqrt{7}}{2}\mathbf{i}$.
So $y_h = C_1 e^{t/2} \cos((\sqrt{7}/2)t) + C_2 e^{t/2} \sin((\sqrt{7}/2)t)$.
(0 - c) Characteristic equation
$4r^2 - 4r + 1 = (2r-1)(2r-1) = 0$ with roots $\frac{1}{2}, \frac{1}{2}$
So $y_h = C_1 e^{t/2} + C_2 t e^{t/2}$.

(1) $[t^2 + 1]$ associated root 0.

$7te^{2t}$ associated root 2.

$-e^{t/2}\sin((\sqrt{7}/2)t)$ associated roots $\frac{1}{2} \pm \frac{\sqrt{7}}{2}\mathbf{i}$.

$-t^2e^{t/2}$ associated root $\frac{1}{2}$.

$$
\begin{aligned}
Y^1(t) &= [At^2 + Bt + C] \\
&+[(Dt + E)e^{2t}] \\
&+[Fe^{t/2}\cos((\sqrt{7}/2)t) + Ge^{t/2}\sin((\sqrt{7}/2)t)] \\
&+[(Ht^2 + It + J)e^{t/2}].
\end{aligned}
$$

For equation (a) the root 2 is a root of the homogeneous equation and so
$Y(t) = [At^2 + Bt + C] + t[(Dt + E)e^{2t}]$
$+[Fe^{t/2}\cos((\sqrt{7}/2)t) + Ge^{t/2}\sin((\sqrt{7}/2)t)] + [(Ht^2 + It + J)e^{t/2}]$.

For equation (b) the roots $\frac{1}{2} \pm \frac{\sqrt{7}}{2}\mathbf{i}$. are roots of the homogeneous equation and so
$Y(t) = [At^2 + Bt + C] + [(Dt + E)e^{2t}]$
$+t[Fe^{t/2}\cos((\sqrt{7}/2)t) + Ge^{t/2}\sin((\sqrt{7}/2)t)] + [(Ht^2 + It + J)e^{t/2}]$.

For equation (c) the root $\frac{1}{2}$ is a repeated root of the homogeneous equation and so
$Y(t) = [At^2 + Bt + C] + [(Dt + E)e^{2t}]$
$+[Fe^{t/2}\cos((\sqrt{7}/2)t) + Ge^{t/2}\sin((\sqrt{7}/2)t)] + t^2[(Ht^2 + It + J)e^{t/2}]$.

Exercise 3.5/ BD19; BDM14 :
$y'' + 4y = 3\sin 2t, \qquad y(0) = 2, y'(0) = -1.$
STEP 0: Roots for the homogeneous are $\pm 2\mathbf{i}$. So
$y_h = C_1 \cos(2t) + C_2 \sin(2t)$.
STEP 1: Associated root for $3\sin 2t$ is $\pm 2\mathbf{i}$ and so STEP 2:
$Y_p^1 = A\cos(2t) + B\sin(2t)$.
STEP 3: $Y_p = At\cos(2t) + Bt\sin(2t)$, because the associated roots are roots of the homogeneous equation.
STEP 4: Substitute

$$
\begin{array}{rl}
4\times & Y_p = At\cos(2t) + Bt\sin(2t) \\
0\times & Y_p' = 2Bt\cos(2t) - 2At\sin(2t) + A\cos(2t) + B\sin(2t) \\
\underline{1\times} & \underline{Y_p'' = -4At\cos(2t) - 4Bt\sin(2t) + 4B\cos(2t) - 4A\sin(2t)} \\
3\sin 2t = & \quad 0 \quad + \quad 0 \quad + \quad 4B\cos(2t) - 4A\sin(2t).
\end{array}
$$

So $B = 0, A = -3/4$ and
$y_g = C_1\cos(2t) + C_2\sin(2t) - \frac{3}{4}t\cos(2t)$.

Now for the initial conditions:

$$y_g = C_1 \cos(2t) + C_2 \sin(2t) - \frac{3}{4} t \cos(2t) + 0t \sin(2t),$$

$$y_g' = (2C_2 - \frac{3}{4}) \cos(2t) - 2C_1 \sin(2t) + 0t \cos(2t) + \frac{3}{4} t \sin(2t).$$

Substitute $t = 0$.

$$\begin{aligned} 2 &= y_g(0) = C_1, \\ -1 &= y_g'(0) = 2C_2 - \frac{3}{4}. \end{aligned}$$

So $C_1 = 2, C_2 = -\frac{1}{8}$ and

$$y = 2 \cos(2t) - \frac{1}{8} \sin(2t) - \frac{3}{4} t \cos(2t).$$

Illustration

As an illustration of the Method of Undetermined Coefficients, we consider the integral problems from Calculus 2:

$$\int e^{ax} \cos bx \; dx, \qquad \int e^{ax} \sin bx \; dx.$$

These were done there by a looping pair of integrations by parts. Instead think of these are the solutions to the first order linear equations with constant coefficients:

$$y' = e^{ax} \cos bx, \qquad y' = e^{ax} \sin bx, \quad (b \neq 0)$$

which we will solve using the Method of Undetermined Coefficients

$$y = Ae^{ax}\cos bx + Be^{ax}\sin bx$$
$$y' = (Aa + Bb)e^{ax}\cos bx + (-Ab + Ba)e^{ax}\sin bx.$$

Equating coefficients between y' and $1e^{ax}\cos bx$ we get

$$\begin{aligned} Aa &+ Bb = 1 \\ -Ab &+ Ba = 0. \end{aligned}$$

Solve using Cramer's Rule with coefficient determinant $a^2 + b^2$

$A = a/(a^2 + b^2), B = b/(a^2 + b^2)$ and so

$$\int e^{ax} \cos bx dx = \frac{ae^{ax} \cos bx \ + \ be^{ax} \sin bx}{a^2 + b^2}$$

Similarly, using

$$\begin{aligned} Aa \ + \ Bb \ &= \ 0 \\ -Ab \ + \ Ba \ &= \ 1, \end{aligned}$$

we obtain

$$\int e^{ax} \sin bx dx = \frac{-be^{ax} \cos bx + ae^{ax} \sin bx}{a^2 + b^2}.$$

For those who have had linear algebra:
The functions $\{e^{ax}\cos bx, e^{ax}\sin bx\}$ form a basis for a two dimensional vector space of functions.
On it the derivative map D is a linear map with matrix

$$D = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

That is, the first and second columns are the coefficients of D applied to $e^{ax}\cos bx$ and $e^{ax}\sin bx$, respectively.

The inverse of D is the integral map I with inverse matrix:

$$I = \frac{1}{a^2 + b^2} \cdot \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

with the first and second columns the coefficients of I applied to $e^{ax} \cos bx$ and $e^{ax} \sin bx$, respectively.

Higher Order, Homogeneous, Linear Equations with Constant Coefficients, Section 4.2

For $n = 3, 4, \ldots$ an nth order linear equation with constant coefficients is of the form

$$A_n y^{(n)} + A_{n-1} y^{(n-1)} + \cdots + A_1 y' + A_0 y = f.$$

It is homogeneous when $f = 0$. We will consider just the homogeneous case and the cases with associated roots so that the method of Undetermined Coefficients can be used.

For an nth order equation the general solution will have n arbitrary constants. In the linear case, we look for n independent solutions $y_1, \ldots, y_n$ and get the general solution of the homogeneous equation $y_h = C_1 y_1 + \cdots + C_n y_n$.

As in the second order case, we look for solutions of the form e^{rt}. When we substitute and divide away the common factor of e^{rt} we obtain the nth degree characteristic equation $P(r) = 0$. with

$$P(r) = A_n r^n + A_{n-1} r^{n-1} + \cdots + A_1 r + A_0.$$

The equation $P(r) = 0$ has, in general, n roots obtained by factoring. There are two CASES.

Real roots ($r = a$): The root $r = a$ from the factor $r - a$ gives a solution e^{at}. "Repeated roots" really means repeated factors. If we have $(r - a)^k$ in the characteristic polynomial $P(r)$, that is, k copies of $(r - a)$ then, as in the second order case, the additional solutions are obtained by multiplying by t, to get solutions $e^{at}, te^{at}, \ldots, t^{k-1}e^{at}$.

Complex conjugate pairs ($r = a \pm b\mathbf{i}$): The conjugate pair $r = a \pm b\mathbf{i}$ comes from a quadratic factor $r^2 - 2ar + (a^2 + b^2)$ gives the two solutions $e^{at}\cos(bt)$ and $e^{at}\sin(bt)$. But now you can have repeated complex roots. If $(r^2 - 2ar + (a^2 + b^2))^k$ occurs in the factoring of $P(r)$ then we have $2k$ solutions $e^{at}\cos(bt), te^{at}\cos(bt), \ldots, t^{k-1}e^{at}\cos(bt)$ and $e^{at}\sin(bt), te^{at}\sin(bt), \ldots, t^{k-1}e^{at}\sin(bt)$.

The hard work here is doing the factoring. So we have to review (or for some of you introduce) some factoring methods.

Example A: $y^{(11)} - 2y^{(7)} + y^{(3)} = 0$ with

$$P(r) = r^{11} - 2r^7 + r^3.$$

First, pull out the common factor $P(r) = r^3(r^8 - 2r^4 + 1)$.

Next notice that $r^8 - 2r^4 + 1$ is quadratic in r^4 and factors to $(r^4 - 1)^2$. Now use the *difference of two squares* $r^4 - 1 = (r^2 + 1)(r^2 - 1) = (r^2 + 1)(r + 1)(r - 1)$. So

$$P(r) = r^3(r^2 + 1)^2(r + 1)^2(r - 1)^2$$

with eleven roots $\quad 0, 0, 0, \pm\mathbf{i}, \pm\mathbf{i}, -1, -1, +1, +1.$

$$y_h = C_1 + C_2 t + C_3 t^2 + C_4 \cos(t) + C_5 t \cos(t) + C_6 \sin(t) + C_7 t \sin(t) + C_8 e^{-t} + C_9 t e^{-t} + C_{10} e^t + C_{11} t e^t.$$

Example B: $y^{(5)} - 3y^{(3)} - 8y'' + 24y = 0$ with

$$P(r) = r^5 - 3r^3 - 8r^2 + 24 = (r^2 - 3)r^3 - (r^2 - 3)8 = (r^2 - 3)(r^3 - 8).$$

This is *factoring by grouping*. The roots of $r^2 - 3 = 0$ are $\pm\sqrt{3}$. For the cubic we need the *difference of two cubes*

$$r^3 - a^3 = (r - a)(r^2 + ar + a^2),$$
$$r^3 + a^3 = r^3 - (-a)^3 = (r + a)(r^2 - ar + a^2).$$

So $P(r) = (r - \sqrt{3})(r + \sqrt{3})(r - 2)(r^2 + 2r + 4)$ with roots $\sqrt{3}, -\sqrt{3}, 2, -1 \pm \sqrt{3}\mathbf{i}$ and the solution is

$$y_h = C_1 e^{\sqrt{3}t} + C_2 e^{-\sqrt{3}t} + C_3 e^{2t} + C_4 e^{-t}\cos(\sqrt{3}t) + C_5 e^{-t}\sin(\sqrt{3}t).$$

DeMoivre's Theorem

Remember that if we start with a complex number z in rectangular form $z = x + \mathbf{i}y$ and convert to polar coordinates with $x = r\cos(\theta), y = r\sin(\theta)$. The length r is called the *magnitude* and the angle θ is called the *argument* of the complex number. So we have

$$z \quad = \quad x + \mathbf{i}y \quad = \quad r(\cos(\theta) + \mathbf{i}\sin(\theta)) \quad = \quad re^{\mathbf{i}\theta}.$$

If $z = re^{\mathbf{i}\theta}$ and $w = ae^{\mathbf{i}\phi}$ then, by using properties of the exponential, we see that $zw = rae^{\mathbf{i}(\theta+\phi)}$. That is, the magnitudes multiply and the arguments add. In particular, for $n = 1, 2, 3 \ldots$, we see that $z^n = r^n e^{\mathbf{i}n\theta}$.

DeMoivre's Theorem describes the solutions of the equation $z^n = a$ which has n distinct solutions when a is nonzero. We describe these solutions when a is a nonzero real number.

Notice that $re^{\mathbf{i}\theta} = re^{\mathbf{i}(\theta+2\pi)}$. When we raise this equation to the power n the angles are replaced by $n\theta$ and $n\theta + n2\pi$ and so they differ by a multiple of π. But when we divide by n we get different angles. So if $a > 0$, then $a = ae^{\mathbf{i}0}$ and $-a = ae^{\mathbf{i}\pi}$.

$$z^n = a \;\Rightarrow\; z = a^{1/n} \cdot \left\{e^{\mathbf{i}0}, e^{\mathbf{i}2\pi/n}, e^{\mathbf{i}4\pi/n}, .., e^{\mathbf{i}(n-1)2\pi/n}\right\}$$

$$z^n = -a \Rightarrow z = a^{1/n} \cdot \left\{e^{\mathbf{i}\pi/n}, e^{\mathbf{i}(\pi+2\pi)/n}, e^{\mathbf{i}(\pi+4\pi)/n}, .., e^{\mathbf{i}(\pi+(n-1)2\pi)/n}\right\}$$

For $z^n = a$ we start at angle 0 and go around the circle of radius $a^{1/n}$ with n equal steps of $2\pi/n$ radians for each step. For $z^n = -a$ we again go around the circle with n equal steps but this time starting a half step up at angle π/n.

Example C: $y^{(6)} - 64y = 0$, with $P(r) = r^6 - 2^6$. By DeMoivre's Theorem the roots are $2\{1, e^{\mathbf{i}\pi/3}, e^{\mathbf{i}2\pi/3}, -1, e^{\mathbf{i}4\pi/3}, e^{\mathbf{i}5\pi/3}\}$.

$2e^{\mathbf{i}\pi/3}, 2e^{\mathbf{i}5\pi/3}$ is the conjugate pair $1 \pm \mathbf{i}\sqrt{3}$ and $2e^{\mathbf{i}2\pi/3}, 2e^{\mathbf{i}4\pi/3}$ is the conjugate pair $-1 \pm \mathbf{i}\sqrt{3}$.

So the roots of $P(r) = 0$ are $2, -2, 1 \pm \mathbf{i}\sqrt{3}, -1 \pm \mathbf{i}\sqrt{3}$.

In this case, one can also factor using the difference of two squares and difference of two cubes:

$$r^6 - 64 = (r^3 - 8)(r^3 + 8) = (r-2)(r^2+2r+4)(r+2)(r^2-2r+4),$$

$$y_h = C_1 e^{2t} + C_2 e^{-2t} + C_3 e^t \cos(\sqrt{3}t) + C_4 e^t \sin(\sqrt{3}t) + C_5 e^{-t} \cos(\sqrt{3}t) + C_6 e^{-t} \sin(\sqrt{3}t)$$

Example D: $y^{(6)} + 81y'' = 0$ with
$P(r) = r^6 + 81r^2 = r^2(r^4 + 81)$.

By DeMoivre's Theorem the roots of $r^4 + 3^4 = 0$ are

$$3\{e^{\mathbf{i}\pi/4}, e^{\mathbf{i}3\pi/4}, e^{\mathbf{i}5\pi/4}, e^{\mathbf{i}7\pi/4}\}.$$

$3e^{\mathbf{i}\pi/4}, 3e^{\mathbf{i}7\pi/4}$ is the conjugate pair $\frac{3\sqrt{2}}{2} \pm \mathbf{i}\frac{3\sqrt{2}}{2}$.
$3e^{\mathbf{i}3\pi/4}, 3e^{\mathbf{i}5\pi/4}$ is the conjugate pair $\frac{-3\sqrt{2}}{2} \pm \mathbf{i}\frac{3\sqrt{2}}{2}$.
So

$$\begin{aligned} y_h = C_1 + C_2 t + \\ C_3 e^{\frac{3\sqrt{2}t}{2}} \cos(\frac{3\sqrt{2}t}{2}) + C_4 e^{\frac{3\sqrt{2}t}{2}} \sin(\frac{3\sqrt{2}t}{2}) \\ + C_5 e^{-\frac{3\sqrt{2}t}{2}} \cos(\frac{3\sqrt{2}t}{2}) + C_6 e^{-\frac{3\sqrt{2}t}{2}} \sin(\frac{3\sqrt{2}t}{2}). \end{aligned}$$

Undetermined Coefficients for Higher Order Equations, Section 4.3

Example E:

$$y^{(11)} + 32y^{(7)} + 256y^{(3)} =$$
$$3t^2 - 7te^{\sqrt{2}t} + e^{-\sqrt{2}t}\sin(\sqrt{2}t) + t^2e^{\sqrt{2}t}\cos(\sqrt{2}t).$$

$$P(r) = r^{11} + 32r^7 + 256r^3 = r^3(r^4 + 16)^2$$

with roots
$0, 0, 0, \sqrt{2} \pm \sqrt{2}\mathbf{i}, \sqrt{2} \pm \sqrt{2}\mathbf{i}, -\sqrt{2} \pm \sqrt{2}\mathbf{i}, -\sqrt{2} \pm \sqrt{2}\mathbf{i}.$

$$y_h = C_1 + C_2t + C_3t^2$$
$$C_4e^{\sqrt{2}t}\cos(\sqrt{2}t) + C_5e^{\sqrt{2}t}\sin(\sqrt{2}t) +$$
$$C_6te^{\sqrt{2}t}\cos(\sqrt{2}t) + C_7te^{\sqrt{2}t}\sin(\sqrt{2}t)$$
$$C_8e^{-\sqrt{2}t}\cos(\sqrt{2}t) + C_9e^{-\sqrt{2}t}\sin(\sqrt{2}t) +$$
$$C_{10}te^{-\sqrt{2}t}\cos(\sqrt{2}t) + C_{11}te^{\sqrt{2}t}\sin(\sqrt{2}t).$$

The associated roots for the four terms of the forcing function are $0, \sqrt{2}, -\sqrt{2} \pm \sqrt{2}\mathbf{i}, \sqrt{2} \pm \sqrt{2}\mathbf{i}$.

The first version of the test function is

$$Y_p^1 = [At^2 + Bt + C] + [(Dt + E)e^{\sqrt{2}t}] + [Fe^{-\sqrt{2}t}\cos(\sqrt{2}t) + Ge^{-\sqrt{2}t}\sin(\sqrt{2}t)] + [(Ht^2 + It + J)e^{\sqrt{2}t}\cos(\sqrt{2}t) + (Kt^2 + Lt + M)e^{\sqrt{2}t}\sin(\sqrt{2}t)].$$

From the interference with the roots of the homogeneous equation, we must use as the correct test function:

$$Y_p = t^3[At^2 + Bt + C] + [(Dt + E)e^{\sqrt{2}t}] + t^2[Fe^{-\sqrt{2}t}\cos(\sqrt{2}t) + Ge^{-\sqrt{2}t}\sin(\sqrt{2}t)] + t^2[(Ht^2 + It + J)e^{\sqrt{2}t}\cos(\sqrt{2}t) + (Kt^2 + Lt + M)e^{\sqrt{2}t}\sin(\sqrt{2}t)].$$

Variation of Parameters, Section 3.6

For a general second order linear equation $\mathcal{L}(y) = y'' + py' + qy = f$, suppose we have two independent solutions y_1, y_2 for the homogeneous equation $\mathcal{L}(y) = 0$. So the general solution of the homogeneous equation is $y_h = C_1 y_1 + C_2 y_2$.

We need a particular solution for the equation with forcing term f. We look for $y_p = u_1 y_1 + u_2 y_2$ with u_1, u_2 nonconstant functions to be determined.

We impose two conditions. First, we want y_p to be a solution of $\mathcal{L}(y) = f$.

Second, when we take the derivative $y_p' = u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2$. Our second condition is $u_1' y_1 + u_2' y_2 = 0$.

So when we substitute we have

$$\begin{array}{rrcl} q \times & y_p & = & u_1 y_1 + u_2 y_2 \\ p \times & y_p' & = & u_1 y_1' + u_2 y_2' \\ \underline{1 \times} & \underline{y_p''} & = & \underline{u_1 y_1''} \underline{+ u_2 y_2'' + u_1' y_1' + u_2' y_2'} \\ & f & = & 0 + 0 + u_1' y_1' + u_2' y_2'. \end{array}$$

So u_1', u_2' are determined by two linear equations

$$\begin{array}{rcrcl} u_1' y_1 & + & u_2' y_2 & = & 0 \\ u_1' y_1' & + & u_2' y_2' & = & f. \end{array}$$

Notice that the determinant of coefficients is the Wronskian and so is nonzero.

Example 3.6/ BD6; BDM5 : $y'' + 9y = 9\sec^2(3t)$. The roots of the homogeneous are $\pm 3\mathbf{i}$ and so $y_1 = \cos(3t), y_2 = \sin(3t)$. We look for $y_p = u_1\cos(3t) + u_2\sin(3t)$. We have the linear equations:

$$\begin{array}{rcrcl} u_1'(\cos(3t)) & + & u_2'(\sin(3t)) & = & 0 \\ u_1'(-3\sin(3t)) & + & u_2'(3\cos(3t)) & = & 9\sec^2(3t). \end{array}$$

The Wronskian is $3(\cos^2(3t) + \sin^2(3t)) = 3$ and so by Cramer's Rule

$$u_1' = -3\sin(3t)\sec^2(3t) = -3\sec(3t)\tan(3t) \;\Rightarrow\; u_1 = -\sec(3t)$$
$$u_2' = 3\cos(3t)\sec^2(3t) = 3\sec(3t) \;\Rightarrow\; u_2 = \ln|\sec(3t) + \tan(3t)|$$

$$y_p = -\sec(3t)\cos(3t) + (\ln|\sec(3t) + \tan(3t)|)\sin(3t) =$$
$$-1 + (\ln|\sec(3t) + \tan(3t)|)\sin(3t).$$

Example 3.6/ BD7; BDM6 : $y'' + 4y' + 4y = t^{-2}e^{-2t}$. The characteristic equation is $r^2 + 4r + 4 = (r+2)^2 = 0$ and so $y_h = C_1 e^{-2t} + C_2 te^{-2t}$. We look for $y_p = u_1 e^{-2t} + u_2 te^{-2t}$. We have the linear equations:

$$\begin{aligned} u_1' e^{-2t} + u_2' te^{-2t} &= 0 \\ u_1'(-2e^{-2t}) + u_2'(-2te^{-2t} + e^{-2t}) &= t^{-2}e^{-2t}. \end{aligned}$$

The Wronskian is e^{-4t} and so

$$\begin{aligned} u_1' = -t^{-1}e^{-4t}/e^{-4t} = -t^{-1} &\Rightarrow \quad u_1 = -\ln(t) \\ u_2' = t^{-2}e^{-4t}/e^{-4t} = t^{-2} &\Rightarrow \quad u_2 = -t^{-1}. \end{aligned}$$

$y_g = C_1 e^{-2t} + C_2 te^{-2t} - \ln(t)e^{-2t} - t^{-1}te^{-2t}$ or

$$y_g = C_1 e^{-2t} + C_2 te^{-2t} - \ln(t)e^{-2t}.$$

Example 3.6/ BD16 : $(1-t)y'' + ty' - y = 2(t-1)^2 e^{-t}$ with $y_1 = e^t, y_2 = t$. Check that y_1 and y_2 are solutions of the homogeneous equation.

Divide by $(1-t)$ to get $r = -2(t-1)e^{-t}$ and so

$$\begin{aligned} u_1' e^t + u_2' t &= 0 \\ u_1' e^t + u_2' &= -2(t-1)e^{-t}. \end{aligned}$$

The Wronskian is $(1-t)e^t$ and so

$$u_1' = 2(t-1)te^{-t}/(1-t)e^t = -2te^{-2t} \Rightarrow \quad u_1 = te^{-2t} + \frac{1}{2}e^{-2t}$$

$$u_2' = -2(t-1)/(1-t)e^t = 2e^{-t} \Rightarrow \quad u_2 = -2e^{-t}.$$

$$y_p = (t + \tfrac{1}{2})e^{-t} - 2te^{-t} = (\tfrac{1}{2} - t)e^{-t}.$$

Spring Problems, Sections 3.7, 3.8

The *restoring force* of a spring is in the opposite direction to its displacement y from equilibrium. In the case of a *linear spring*, which is all we is consider, it is proportional to the displacement and so is $-ky$ with k a positive constant called the *spring constant*.

If there is friction - imagine the spring moving through a liquid like molasses - then the friction force is in the opposite direction to its velocity $v = \frac{dy}{dt} = y'$. We will assume it is proportional to the velocity and so is $-cv$ where c is a positive constant (or zero if there is no friction).

Finally, there may be an external force, called the *forcing term*, $f(t)$ which may not be a constant. It is usually assume periodic.

Thus, the total force is $-ky - cy' + f(t)$. Newton's Third Law says that the total force is proportional to the the acceleration $a = \frac{d^2y}{dt^2} = y''$ and the proportionality constant is the *mass* m. That is, $F = ma$. So the differential equation for the spring is:

$$my'' + cy' + ky = 0.$$

The *weight* w of an object is the force due to gravity. The acceleration due to gravity is a constant (near the surface of the earth) denoted $-g$. (Why the minus sign?) In English units $g = 32 ft/sec^2$. Thus, weight and mass are related by $w = mg$.

When a weight is attached to a hanging spring it stretches it a distance denoted ΔL. The equilibrium position is when the spring force exactly balances the weight. That is, $w = k\Delta L$. So the equations which relate the constants in these problems are

$$w = mg, \qquad w = k\Delta L.$$

Suppose there is no friction and no external force so that $my'' + ky = 0$. This is a second order, linear, homogeneous equation with constant coefficients. The characteristic equation is $mr^2 + k = 0$ with roots $\pm\omega\mathbf{i}$ where $\omega = \sqrt{k/m} = \sqrt{g/\Delta L}$, the *natural frequency* of the spring. Thus, the general solution is

$$y = C_1 \cos(\omega t) + C_2 \sin(\omega t),$$

with the constants $C_1 = y(0)$, the initial position, and $\omega C_2 = y'(0)$, the initial velocity.

Regarding the pair (C_1, C_2) as a point in the plane we can write it as $(A\cos(\phi), A\sin(\phi))$, with $A^2 = C_1^2 + C_2^2$. Remember that $\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$ and so

$$y = A\cos(\omega t)\cos(\phi) + A\sin(\omega t)\sin(\phi) = A\cos(\omega t - \phi).$$

If there is damping, and so $c > 0$, then the characteristic equation is $r^2 + 2(c/2m)r + (k/m) = 0$
If the damping is small, with $(c/2m) < \sqrt{k/m}$ then the roots are a complex conjugate pair $-(c/2m) \pm \sqrt{(k/m) - (c/2m)^2}\mathbf{i}$ and the solution is

$$y = e^{-(c/2m)t}[C_1 \cos(\sqrt{(k/m) - (c/2m)^2}\ t) \quad + \\ C_2 \sin(\sqrt{(k/m) - (c/2m)^2}\ t)].$$

If the damping is large, with $(c/2m) > \sqrt{k/m}$ then the roots are real with

$$y = C_1 e^{[-(c/2m)+\sqrt{(c/2m)^2-(k/m)}]\ t} + C_2 e^{[-(c/2m)-\sqrt{(c/2m)^2-(k/m)}]\ t}.$$

Finally, if $(c/2m) = \sqrt{k/m}$ then $-(c/2m)$ is a repeated root and so $y = e^{-(c/2m)t}[C_1 + C_2 t]$.
In any case, the solution decays exponentially.

In the undamped case, suppose there is periodic forcing: $y'' + \omega_0^2 y = \cos(\omega t)$.

The roots of the characteristic equation $r^2 + \omega_0^2 = 0$ are $\pm\omega_0\mathbf{i}$. ω_0 is the *natural frequency* of the spring.

The associated roots for the forcing term are $\pm\omega\mathbf{i}$.

If $\omega \neq \omega_0$ then the test function $Y_p = A\cos(\omega t) + B\sin(\omega t)$.

If $\omega = \omega_0$ then the test function $Y_p = At\cos(\omega t) + Bt\sin(\omega t)$. Forcing at the natural frequency of the spring is called *resonance*.

Example 3.8/ BD11; BDM7 : A spring is stretched $6in$ by a mass that weighs $4lb$. The mass is attached to a dashpot with a damping constant of $0.25lb \cdot sec/ft$ and is acted upon by an external force of $4\cos(2t)lb$.

- Determine the general solution and the steady state response of the system.
- If the given mass is replaced by one of mass m, determine m so that the amplitude of the steady state response is a maximum.

$\Delta L = \frac{1}{2}, w = 4, c = \frac{1}{4}, g = 32$, and so
$k = w/\Delta L = 8, m = w/g = \frac{1}{8}$. So

$$\frac{1}{8}y'' + \frac{1}{4}y' + 8y = 4\cos(2t), \quad \text{or} \quad y'' + 2y' + 64y = 32\cos(2t).$$

$$y_h = e^{-t}[C_1 \cos(\sqrt{63}\ t) \ + \ C_2 \sin(\sqrt{63}\ t)]$$

The particular solution $y_p = A\cos(2t) + B\sin(2t)$ is the steady-state solution.

Substitute into the equation $y'' + 2y' + 64y = 32\cos(2t)$

$$\begin{array}{rrcrcr} 64 \times & [\ y_p & = & A\cos(2t) & + & B\sin(2t)\ ] \\ 2 \times & [\ y_p' & = & 2B\cos(2t) & - & 2A\sin(2t)\ ] \\ \underline{1 \times} & \underline{[\ y_p''} & \underline{=} & \underline{-4A\cos(2t)} & \underline{-} & \underline{4B\sin(2t)\ ]} \\ & 32\cos 2t & = & (60A + 4B)\cos(2t) & + & (-4A + 60B)\sin(2t). \end{array}$$

$$\begin{array}{rcrcr} 15A & + & B & = & 8 \\ -A & + & 15B & = & 0 \end{array} \quad \Rightarrow \quad (A, B) = \frac{4}{113}(15, 1).$$

The amplitude is $\sqrt{A^2 + B^2} = \frac{4\sqrt{226}}{113}$.

If the mass m is substituted then the equation is $my'' + \frac{1}{4}y' + 8y = 4\cos(2t)$. As there is damping the steady-state solution is again $y_p = A\cos(2t) + B\sin(2t)$.

$$
\begin{array}{rcccc}
8 \times [\ y_p & = & A\cos(2t) & + & B\sin(2t)\ ] \\
\frac{1}{4} \times [y_p' & = & 2Bt\cos(2t) & - & 2At\sin(2t)\ ] \\
\underline{m \times [\ y_p''} & = & -4A\cos(2t) & - & 4B\sin(2t)\ ]
\end{array}
$$

$$
\begin{aligned}
4\cos 2t &= ((-4m+8)A + \frac{1}{2}B)\cos(2t) \\
&\quad + (-\frac{1}{2}A + (-4m+8)B)\sin(2t).
\end{aligned}
$$

$$\begin{array}{ccccc} (-8m+16)A & + & B & = & 8 \\ -A & + & (-8m+16)B & = & 0 \end{array}$$

The coefficient determinant is $64(2-m)^2+1$ and so

$$(A,B) = \frac{8}{64(2-m)^2+1}(8(2-m),1).$$

The amplitude is $\frac{8}{\sqrt{64(2-m)^2+1}}$ which has its maximum at $m=2$.

Series Solutions Centered at an Ordinary Point

We are going to look at second order, linear, homogeneous equations of the form $P(x)y'' + Q(x)y' + R(x)y = 0$ with the coefficients P, Q, R polynomials in x. The center $x = a$ is an *ordinary point* when $P(a)$ is not zero so that Q/P and R/P are defined at $x = a$.

We will look for a series solution centered at $x = 0$. Thus,

$$\begin{aligned} y &= \sum a_n x^n. \\ y' &= \sum n a_n x^{n-1}. \\ y'' &= \sum n(n-1) a_n x^{n-2}. \end{aligned}$$

We use three steps

Step 1: Write a separate series for each term of the polynomials P, Q and R.

Step 2: Shift indices so that each series is represented as powers of x^k.

Step 3: Collect terms to obtain the *recursion formula* for the a_k's with $a_0 = y(0)$ and $a_1 = y'(0)$ the arbitrary constants.

Example: $(2+3x^2)y'' + (x-5x^3)y' + (1-x^2+x^4)y = 0$. So there will be seven series.

$$
\begin{aligned}
2y'' &= \Sigma\ 2n(n-1)a_n x^{n-2}[k=n-2] = \Sigma 2(k+2)(k+1)a_{k+2}x^k. \\
3x^2y'' &= \Sigma\ 3n(n-1)a_n x^n\ [k=n] = \Sigma\ 3k(k-1)a_k x^k. \\
xy' &= \Sigma\ na_n x^n\ \ [k=n] = \Sigma\ ka_k x^k. \\
-5x^3y' &= \Sigma -5na_n x^{n+2}[k=n+2] = \Sigma -5(k-2)a_{k-2}x^k. \\
1y &= \Sigma\ a_n x^n\ \ [k=n] = \Sigma\ a_k x^k. \\
-x^2y &= \Sigma -a_n x^{n+2}[k=n+2] = \Sigma\ -a_{k-2}x^k. \\
x^4y &= \Sigma\ a_n x^{n+4}\ \ [k=n+4] = \Sigma\ a_{k-4}x^k.
\end{aligned}
$$

We sum and collect the coefficients of x^k to get the recursion formula.

Recursion Formula:

$$a_{k+2} = \frac{1}{2(k+2)(k+1)}[-3k(k-1)a_k$$
$$-ka_k + 5(k-2)a_{k-2} - a_k + a_{k-2} - a_{k-4}] =$$
$$\frac{1}{2(k+2)(k+1)}[(-k(3k-2)-1)a_k + (5k-9)a_{k-2} - a_{k-4}].$$

Substitute:

$k = 0, \quad a_2 = \frac{1}{4}[-a_0] = -\frac{1}{4}a_0.$

$k = 1, \quad a_3 = \frac{1}{12}[-2a_1] = -\frac{1}{6}a_1.$

$k = 2, \quad a_4 = \frac{1}{24}[-9a_2 + 1a_0] = \frac{1}{24}[\frac{13}{4}a_0] = \frac{13}{96}a_0.$

$k = 3, \quad a_5 = \frac{1}{40}[-22a_3 + 6a_1] = \frac{1}{40}[\frac{29}{3}a_1] = \frac{29}{120}a_1.$

$k = 4, \quad a_6 = \frac{1}{60}[-41a_4 + 11a_2 - a_0] =$
$\frac{1}{60}[-\frac{41 \cdot 13}{96}a_0 - \frac{11}{4}a_0 - a_0] = -\frac{893}{5760}a_0.$

Example: $(1-x)y''+(x-2x^2)y'+(1-x+x^2)y=0.y(0)=-12, y'(0)=12$ So there will again be seven series. Step 1: and Step 2:

$$
\begin{aligned}
1y'' &= \Sigma\ 1n(n-1)a_n x^{n-2}[k=n-2] = \Sigma 1(k+2)(k+1)a_{k+2}x \\
-xy'' &= \Sigma\ -n(n-1)a_n x^{n-1}\ [k=n-1] = \Sigma\ -(k+1)(k)a_{k+1}x^k. \\
xy' &= \Sigma\ na_n x^n\ \ [k=n] = \Sigma\ ka_k x^k. \\
-2x^2y' &= \Sigma -2na_n x^{n+1}[k=n+1] = \Sigma -2(k-1)a_{k-1}x^k. \\
1y &= \Sigma\ a_n x^n\ \ [k=n] = \Sigma\ a_k x^k. \\
-xy &= \Sigma -a_n x^{n+1}[k=n+1] = \Sigma\ -a_{k-1}x^k. \\
x^2y &= \Sigma\ a_n x^{n+2}\ \ [k=n+2] = \Sigma\ a_{k-2}x^k.
\end{aligned}
$$

We sum and collect the coefficients of x^k to get the recursion formula.

Recursion Formula:

$$a_{k+2} = \frac{1}{(k+2)(k+1)}[(k+1)(k)a_{k+1} - ka_k + 2(k-1)a_{k-1} - a_k + a_{k-1} - a_{k-2}] =$$

$$\frac{1}{(k+2)(k+1)}[((k+1)(k)a_{k+1} - (k+1)a_k + (2k-1)a_{k-1} - a_{k-2}].$$

Substitute: $a_0 = -12, a_1 = 12$

$k = 0, \quad a_2 = \frac{1}{2}[0 - a_0 + 0 + 0] = 6.$

$k = 1, \quad a_3 = \frac{1}{6}[2a_2 - 2a_1 + a_0 + 0] = -4.$

$k = 2, \quad a_4 = \frac{1}{12}[6a_3 - 3a_2 + 3a_1 - a_0] = \frac{1}{2}.$

$k = 3, \quad a_5 = \frac{1}{20}[12a_4 - 4a_3 + 5a_2 - a_1] = 2.$

$k = 4, \quad a_6 = \frac{1}{30}[20a_5 - 5a_4 + 7a_3 - a_2] = \frac{7}{60}$

$$y = -12 + 12x + 6x^2 - 4x^3 + \frac{1}{2}x^4 + 2x^5 + \frac{7}{60}x^6 + \ldots$$

Example: $y'' + [2 + (x-1)^2]y' + (x^2-1)y = 0$ centered at $x = 1$ So we want a solution of the form $y = \Sigma a_n (x-1)^n$. Let $X = x - 1$ so that $x = X + 1$. Then $x^2 - 1 = X^2 + 2X$ and $y = \Sigma a_n X^n$ with $y'' + [2 + X^2]y' + (X^2 + 2X)y = 0$. Notice that $\frac{dy}{dX} = \frac{dy}{dx}$ because $\frac{dX}{dx} = 1$.

$$
\begin{aligned}
y'' &= \Sigma\ n(n-1)a_n X^{n-2} & [k = n-2] &= \Sigma (k+2)(k+1)a_{k+2}X^k. \\
2y' &= \Sigma\ 2na_n X^{n-1} & [k = n-1] &= \Sigma\ 2(k+1)a_{k+1}X^k. \\
X^2y' &= \Sigma\ na_n X^{n+1} & [k = n+1] &= \Sigma\ (k-1)a_{k-1}X^k. \\
X^2y &= \Sigma\ a_n X^{n+2} & [k = n+2] &= \Sigma\ a_{k-2}X^k. \\
2Xy &= \Sigma\ 2a_n X^{n+1} & [k = n+1] &= \Sigma\ 2a_{k-1}X^k.
\end{aligned}
$$

Recursion Formula:

$$a_{k+2} = \frac{1}{(k+2)(k+1)}[-2(k+1)a_{k+1} - (k+1)a_{k-1} - a_{k-2}].$$

From the recursion formula
$a_{k+2} = \frac{1}{(k+2)(k+1)}[-2(k+1)a_{k+1} - (k+1)a_{k-1} - a_{k-2}]$, we substitute to get $k = 0, \quad a_2 = \frac{1}{2}[-2a_1 - 0 - 0] = -a_1$.

$k = 1, \quad a_3 = \frac{1}{6}[-4a_2 - 2a_0 - 0] = \frac{2}{3}a_1 - \frac{1}{3}a_0.$
$k = 2, \quad a_4 = \frac{1}{12}[-6a_3 - 3a_1 - a_0] = -\frac{7}{12}a_1 + \frac{1}{12}a_0$
$k = 3, \quad a_5 = \frac{1}{20}[-8a_4 - 4a_2 - a_1] = \frac{23}{60}a_1 - \frac{1}{30}a_0.$
$k = 4, \quad a_6 = \frac{1}{30}[-10a_5 - 5a_3 - a_2] = -\frac{37}{180}a_1 + \frac{1}{15}a_0.$

$a_0 = y(1)$ and $a_1 = y'(1)$.

If the initial conditions $y(1) = 18, y'(1) = -24$ are given, it is best to substitute immediately. From the recursion formula $a_{k+2} = \frac{1}{(k+2)(k+1)}[-2(k+1)a_{k+1} - (k+1)a_{k-1} - a_{k-2}]$, we substitute to get

$$
\begin{aligned}
k = 0, \quad & a_2 = \tfrac{1}{2}[48 - 0 - 0] = 24. \\
k = 1, \quad & a_3 = \tfrac{1}{6}[-96 - 36 - 0] = -22. \\
k = 2, \quad & a_4 = \tfrac{1}{12}[132 + 72 - 18] = \tfrac{31}{2}. \\
k = 3, \quad & a_5 = \tfrac{1}{20}[-124 - 96 + 24] = -\tfrac{49}{5}. \\
k = 4, \quad & a_6 = \tfrac{1}{30}[98 + 110 - 24] = \tfrac{92}{15}.
\end{aligned}
$$

So that

$$
\begin{aligned}
y = 18 - 24(x-1) + 24(x-1)^2 - 22(x-1)^3 + \frac{31}{2}(x-1)^4 \\
- \frac{49}{5}(x-1)^5 + \frac{92}{15}(x-1)^6 + \ldots .
\end{aligned}
$$