

(1) (12 points each) Compute the general solution of each of the following differential equations:

$$(a) \quad y' = \frac{x^2}{y+yx^3}.$$

Variables Separable: $\int y \, dy = \int \frac{x^2}{1+x^3}.$

$$\frac{1}{2}y^2 = \frac{1}{3}\ln(1+x^3) + C.$$

$$(b) \quad xy' + 2y = e^{-x} - xy$$

Linear: $y' + (\frac{2}{x} + 1)y = e^{-x}/x.$

$$\mu = \exp\left(\int \left(\frac{2}{x} + 1\right) dx\right) = \exp(2 \ln x + x) = x^2 e^x.$$

$$(x^2 e^x y)' = x^2 e^x y' + (2x + x^2)e^x y = x.$$

$$x^2 e^x y = \frac{x^2}{2} + C, \quad \text{or} \quad y = \left(\frac{1}{2} + \frac{C}{x^2}\right)e^{-x}.$$

$$(c) \quad y'' - 7y = 0 \quad (y \text{ is a function of } x).$$

Second Order Linear: Characteristic Equation $r^2 - 7 = 0$ with roots $r = \pm\sqrt{7}.$

$$y = C_1 e^{\sqrt{7}x} + C_2 e^{-\sqrt{7}x}.$$

$$(d) (y - x)dy + 2ydx = 0.$$

$$y' = \frac{2y}{x - y}.$$

Homogeneous with $y = xz$:

$$x \frac{dz}{dx} = \frac{2z}{1-z} - z = \frac{z^2+z}{1-z}.$$

$$\int \frac{1-z}{z(z+1)} dz = \int \frac{dx}{x}.$$

$$\frac{1-z}{z(z+1)} = \frac{A}{z} + \frac{B}{(z+1)}.$$

$$1 - z = A(z + 1) + Bz. \quad \text{with } z = 0, A = 1; z = -1, B = -2.$$

$$\ln(z) - 2\ln(z + 1) = \ln(x) + C.$$

$$(y/x)(x/(y+x))^2 = Cx, \quad \text{or} \quad \frac{y}{(y+x)^2} = C.$$

Alternatively, as some students noted, you can rewrite the equation as:

$$2y \frac{dx}{dy} - x = -y, \quad \text{or} \quad \frac{dx}{dy} + \left(-\frac{1}{2y}\right)x = -\frac{1}{2}.$$

This is linear with x as a function of y . $\mu = \exp\left(-\int \frac{dy}{2y}\right) = y^{-1/2}$.

$$(y^{-1/2}x)' = y^{-1/2} \frac{dx}{dy} - \frac{1}{2}y^{-3/2}x = -\frac{1}{2}y^{-1/2}.$$

$$y^{-1/2}x = -y^{1/2} + C \quad \text{or} \quad x = -y + C\sqrt{y}.$$

$$(y+x)^2 = Cy.$$

(2) (12 points each) Solve the following initial value problems:

(a)(a) $y'y'' = 2$, with $y(0) = 1, y'(0) = 2$.

Reduction of Order: Let $y' = v$ so that $y'' = \frac{dv}{dt}$.

$$v \frac{dv}{dt} = 2.$$

$$v^2/2 = \int v dv = \int 2 dt = 2t + C_1.$$

When $t = 0$, $v = 2$ and so $C_1 = 2$, $v^2 = 4(t + 1)$.

$\frac{dy}{dt} = v = 2\sqrt{t + 1}$ (positive square root because $v = 2$ when $t = 0$). So

$$y = 2 \int (1 + t)^{1/2} dt = \frac{4}{3}(t + 1)^{3/2} + C_2.$$

When $t = 0$, $y = 1$ and so $C_2 = -\frac{1}{3}$.

$$y = \frac{4(t + 1)^{3/2} - 1}{3}.$$

(b) $(xy \cos(xy) + \sin(xy) + 2x)dx + (x^2 \cos(xy))dy = 0$, with $y(\pi) = 1$.

$$\frac{\partial}{\partial y}(xy \cos(xy) + \sin(xy) + 2x) = 2x \cos(xy) - x^2 y \sin(xy) = \frac{\partial}{\partial x}(x^2 \cos(xy)).$$

$$F = \int (x^2 \cos(xy)) dy = x \sin(xy) + H(x).$$

$$\frac{\partial F}{\partial x} = xy \cos(xy) + \sin(xy) + H'(x) = xy \cos(xy) + \sin(xy) + 2x.$$

So $H'(x) = 2x$, $H(x) = x^2$. and so $F = x \sin(xy) + x^2$. The solution is

$$x \sin(xy) + x^2 = C. \quad \text{with} \quad y(\pi) = 1, \quad C = \pi^2.$$

(3) (10 points) Choose a value of B so that the equation $(Bx/y + y/x)dx + (\ln x - x^2/y^2)dy = 0$ is exact and then solve the equation.

$$\frac{\partial}{\partial y}(Bx/y + y/x) = -Bx/y^2 + 1/x = \frac{\partial}{\partial x}(\ln x - x^2/y^2) \quad \text{when } B = 2$$

$$F = \int (2x/y + y/x)dx = x^2/y + y \ln x + H(y).$$

$$\frac{\partial F}{\partial y} = -x^2/y^2 + \ln x + H'(y) = \ln x - x^2/y^2.$$

So $H(y) = 0$ and $F = x^2/y + y \ln x$.

$$x^2/y + y \ln x = C.$$

(4)(8 points) (a) Assume that y_1, y_2 are solutions of the equation $y'' + py' + qy = 0$ where p and q are functions of t . Prove Abel's Theorem. That is, show that the Wronskian of $\{y_1, y_2\}$ satisfies a first order differential equation.

$$0 = y_1 y_2' - y_2 y_1'.$$

$$W = y_1 y_2' - y_2 y_1'.$$

$$W' = y_1 y_2'' - y_2 y_1''.$$

Multiply the first row by q , the second row by p , the third row by 1 and add.

Because y_1 and y_2 are solutions we obtain:

$$0 + pW + W' = 0 + 0. \quad \text{That is, } \frac{dW}{dt} + pW = 0 \text{ with solution } W = C \cdot \exp\left(-\int p(t)dt\right).$$

(b) Compute the Wronskian of $\{t, \sin t\}$ and explain why the pair cannot be solutions on any interval which includes 0 of any equation $y'' + py' + qy = 0$. (3)(9 points) Assume that y_1, y_2 are solutions of the equation $y'' + py' + qy = 0$ where p and q are functions of t . The Wronskian W of y_1, y_2 is $W = y_1 y_2' - y_2 y_1'$.

So if $\{y_1, y_2\} = \{t, \sin t\}$, then

$$W = t \cos t - \sin t.$$

This equals 0 when $t = 0$ but is not identically zero and so Abel's Theorem implies that the pair cannot be a solution pair on an interval which includes 0.

(5) (5) (10 points) A 400 gallon tank contains 300 gallons of water in which is dissolved 20 pounds of salt. Starting at time $t = 0$, pure water is pumped into the tank at a rate of 5 gallons per minute. At the same time, the well-stirred mixture is pumped out at the rate of 7 gallons per minute. The process is stopped when the tank is empty

(a) How many minutes will it take to empty the tank?

(b) Set up an initial value problem (differential equation and initial conditions) for the amount $Q(t)$ of salt (in pounds) in the tank at time t until the tank is drained dry and solve the equation to determine $Q(t)$.

$$\frac{dV}{dt} = 5 - 7 \text{ in } gal/min \text{ with } V_0 = 300. \text{ So } V = 300 - 2t.$$

(a) $V = 0$ when $t = 150$.

(b) In *pounds/min*

$$\frac{dQ}{dt} = 5 \cdot 0 - 7 \frac{Q}{V} = -\frac{7}{300 - 2t} Q$$

with $Q_0 = 20$.

$$\ln Q = \int -\frac{7}{300 - 2t} dt = \frac{7}{2} \ln(300 - 2t) + C$$

$$Q = C(300 - 2t)^{7/2}, \quad \text{with } 20 = C 300^{7/2}.$$

$$\text{So } Q = 20 \cdot \left(\frac{300-2t}{300}\right)^{7/2} = 20 \cdot \left(\frac{150-t}{150}\right)^{7/2} \text{ for } t \leq 150.$$

Remember to show your work. Good luck.