MATH 391 Test 2 Solutions Fall, 2026

(1) (16 points) Compute the general solution of the homogeneous equation $y^{(8)} - 2y^{(4)} + y = 0$.

The characteristic equation is

$$r^8 - 2r^4 + 1 = (r^4 - 1)^2 = (r^2 - 1)^2(r^2 + 1)^2.$$

So the roots are $1, 1, -1, -1, \pm\mathbf{i}, \pm\mathbf{i}$. The solution is

$$\begin{aligned} y \;=\; & C_1e^t + C_2te^t + C_3e^{-t} + C_4te^{-t} + \\ & C_5\cos(t) + C_6t\cos(t) + C_7\sin(t) + C_8t\sin(t). \end{aligned}$$

(2)(16 points) (a) For the differential equation $y'' - 5y' = 0$ obtain a fundamental pair of solutions and compute the Wronskian of the pair.

(b) Solve the initial value problem: $y'' - 5y' = 100e^{-5t} + 20$ with $y(0) = 0$ and $y'(0) = -14$.

(a) The characteristic polynomial for the homogeneous equation is $r^2 - 5r = r(r-5)$ with roots $0, 5$. The fundamental pair is $\{1 = e^{0t}, e^{5t}\}$ and so the Wronskian $= y_1y_2' - y_2y_1' = 1 \cdot (5e^{5t}) - (e^{5t}) \cdot 0 = 5e^{5t}$.

(b) The general solution of the homogeneous system is $y_h = C_1 + C_2e^{5t}$.

The associated root for $100e^{-5t}$ is -5 and the associated root for 20 is 0. So the first version of the test function is $Y_p^1 = Ae^{-5t} + B$. Because 0 is a root of the homogeneous equation, the test function is $Y_p = Ae^{-5t} + Bt$.

$$\begin{aligned} -5 \times \qquad Y_p' \;&=\; 25Ae^{-5t} \;-\; 5B \\ 1 \times \qquad Y_p'' \;&=\; 25Ae^{-5t} \end{aligned}$$

$$100e^{-5t} + 20 \;=\; 50Ae^{-5t} \;-\; 5B$$

So $A = 2,\ B = -4$.

$$y = C_1 + C_2e^{5t} + 2e^{-5t} - 4t$$
$$y' = +(5C_2)e^{5t} - 10e^{-5t} - 4.$$

Since $y(0) = 0, y'(0) = -14.$, $0 = C_1 + C_2 + 2$ and $-14 = 5C_2 - 14$. So $C_2 = 0, C_1 = -2$.

$$y = -2 + 2e^{-5t} - 4t.$$

(3)(16 points) Consider the sixth order differential equation $y^{(6)} + -16y'' = g(t)$.

(a) Compute the general solution of the homogeneous equation with $g(t) = 0$.

(b) For each of the following forcing functions $g(t)$, write down the test function with the fewest terms which can be used to obtain a particular solution via the Method of Undetermined Coefficients. Do not solve for the constants.

(i) $g(t) = t^2 + e^{2t} + 2.$ (ii) $g(t) = 2e^{2t}\sin(2t) + t\cos(2t).$

(a) The characteristic polynomial is $r^6 - 16r^2 = r^2(r^2 - 4)(r^2 + 4)$. The roots are $0, 0, \pm 2, \pm 2\mathbf{i}$.

$$y_h = C_1 + C_2t + C_3e^{2t} + C_4e^{-2t} + C_5\cos 2t + C_6\sin 2t.$$

(b) (i) Associated root for $t^2 + 2$ is 0 and for e^{2t} is 2. So $Y_p^1 = [At^2 + Bt + C] + [De^t]$ and as 0 is a root of the homogeneous repeated twice and 2 is a root of the homogeneous

$$Y_p = t^2[At^2 + Bt + C] + t[De^{2t}].$$

(ii) Associated roots for $2e^{2t}\sin(2t)$ are $2 \pm 2\mathbf{i}$, and for $t\cos(2t)$ is $\pm 2\mathbf{i}$ So $Y_p^1 = [Ae^{2t}\cos(2t) + Be^{2t}\sin(2t)] + [(Ct + D)\cos(2t) + (Et + F)\sin(2t)]$.

The roots $\pm 2\mathbf{i}$ are roots of the homogeneous, while $2 \pm 2\mathbf{i}$ are not and so

$Y_p = [Ae^{2t}\cos(2t) + Be^{2t}\sin(2t)] + t[(Ct + D)\cos(2t) + (Et + F)\sin(2t)]$.

(4) (10 points) Compute the general solution of the equation $y^{(6)} - y = 0$.

The characteristic equation is $r^6 - 1 = 0$ or $r^6 = 1$. For DeMoivre's Theorem we go around the circle of radius 1 in six equal steps of $60°$ each, starting at 0. Use the $30 - 60 - 90$ special triangle to obtain the roots:

$$1, \frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathbf{i}, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathbf{i}, -1.$$

So the solution is:

$$\begin{aligned} y &= C_1 e^t + C_2 e^{t/2}\cos(\sqrt{3}t/2) + C_3 e^{t/2}\sin(\sqrt{3}t/2) + \\ &\quad C_4 e^{-t/2}\cos(\sqrt{3}t/2) + C_5 e^{-t/2}\sin(\sqrt{3}t/2) + C_6 e^{-t}. \end{aligned}$$

Alternatively, write $r^6 - 1 = (r^3 - 1)(r^3 + 1) = (r-1)(r^2 + r + 1)(r + 1)(r^2 - r + 1)$.

(5)(16 points) Compute the general solution of $y'' + 2y' + y = \frac{2e^{-t}}{1-t^2}$. [Hint: Don't forget the Method of Partial Fractions.]

The characteristic polynomial for the homogeneous equation is $0 = r^2 + 2r + 1 = (r+1)^2$ with roots $-1, -1$. The solution of the homogeneous is $y_h = C_1 e^{-t} + C_2 te^{-t}$. We look for a particular solution $y_p = u_1 e^{-t} + u_2 te^{-t}$.

The equations are:

$$\begin{aligned} u_1' e^{-t} + u_2' te^{-t} &= 0, \\ u_1'(-e^{-t}) + u_2'(e^{-t} - te^{-t}) &= \frac{2e^{-t}}{1-t^2} \end{aligned}$$

The Wronskian is e^{-2t}.

$$\begin{aligned} u_1' &= -\frac{2te^{-2t}}{1-t^2} \div e^{-2t} = -\frac{2t}{1-t^2}, \\ u_2' &= \frac{2e^{-2t}}{1-t^2} \div e^{-2t} = \frac{2}{1-t^2} = \frac{1}{1-t} + \frac{1}{1+t} \end{aligned}$$

by partial fractions.

So $u_1 = \ln(1-t^2), u_2 = \ln\frac{1+t}{1-t}$ and $y_p = e^{-t}\ln(1-t^2) + te^{-t}\ln\frac{1+t}{1-t}$.

$$y_g = C_1 e^{-t} + C_2 te^{-t} + e^{-t}\ln(1-t^2) + te^{-t}\ln\frac{1+t}{1-t}.$$

(6)(16 points)(5)(15 points) For the differential equation, $x^2y'' + 5xy' + 4y = 0$, $y_1 = x^{-2}$ is a solution. Use the method of Reduction of Order to compute a second solution y_2 which is independent of the first one.

Look for $y_2 = ux^{-2}$ So that

$$\begin{array}{rrcl} 4\times & [y_2 & = & ux^{-2}] \\ 5x\times & [y_2' & = & -2ux^{-3} + u'x^{-2}] \\ x^2\times & [y_2'' & = & 6ux^{-4} - 4u'x^{-3} + u''x^{-2}] \end{array}$$

$$0 = 0 + u'x^{-1} + u''.$$

With $v = u'$, $\frac{dv}{v} = -x^{-1}dx$

Therefore, $\ln v = -\ln x$ and so $\frac{du}{dx} = v = x^{-1}$ and $u = \ln x$

$$y_2 = x^{-2}\ln x.$$

(7)(10 points) A hanging spring is stretched 6 inches (= .5 feet) by a weight of 64 pounds.

(a) Set up the initial value problem (differential equation and initial conditions) which describes the motion, neglecting friction, when the weight is pulled down an additional foot and is then released and is subjected to an external force of $6\cos(\omega t)$. You need not solve the equation. (Recall that g, the acceleration due to gravity is 32 feet/second^2.)

(b) Write down the test function with the fewest terms which can be used to obtain a particular solution via the Method of Undetermined Coefficients when $\omega = 4$.

(c) Write down the test function with the fewest terms which can be used to obtain a particular solution via the Method of Undetermined Coefficients when ω is the *resonance frequency* that is, the natural frequency of the spring.

(a) $mg = w = 64$, $m = 2$ $\Delta L = 1/2$ and so $w = k\Delta L$ implies $k = 128$.

The equation is

$$2y'' + 0y' + 128y = 6\cos(\omega t) \text{ or } y'' + 64y = 3\cos(\omega t)$$

with initial conditions $y(0) = -1, y'(0) = 0$.

The characteristic polynomial for the homogeneous equation is r^2+64 with roots $\pm 8\mathbf{i}$. Thus, the natural frequency of the spring is 8.

(b) $Y_p = A\cos(4t) + B\sin(4t)$.

(c) $Y_p = t[A\cos(8t) + B\sin(8t)]$.